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Robust output regulation of minimum phase nonlinear systems using conditional servocompensators

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SUMMARY

We consider the design of a robust continuous sliding mode controller for the output regulation of a class of minimum-phase nonlinear systems. Previous work has shown how to do this by incorporating a linear servocompensator in the sliding mode design, but the transient performance is degraded when compared to ideal sliding mode control. Extending previous ideas from the design of 'conditional integrators' for the case of asymptotically constant references and disturbances, we design the servocompensator as a conditional one that provides servocompensation only inside the boundary layer; achieving asymptotic output regulation, but with improved transient performance. We give both regional as well as semi-global results for error convergence, and show that the controller can be tuned to recover the performance of an ideal sliding mode control. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: output regulation; servomechanisms; output feedback; nonlinear systems; minimum-phase systems

1. INTRODUCTION

The output regulation problem deals with the design of a controller to make the output of a fixed plant asymptotically track (or reject) reference (or disturbance) signals produced by an autonomous system called the *exosystem*. For multivariable, time-invariant, finite-dimensional, linear systems, an exhaustive account of the available theory can be found, for instance, in the works of Davison [1] and Francis and Wonham [2]. Extensions to the nonlinear case have since been studied by many researchers [3–20]. We focus our attention on Reference [17], dealing with a robust servomechanism design for single-input single-output nonlinear systems with well-defined normal form and asymptotically stable zero dynamics. As pointed out in Reference [17], the three basic ingredients of the design, common to earlier results of Khalil and coworkers, are as follows. First, by studying the dynamics of the system on the zero-error manifold, a linear internal model is identified, which generates the trajectories of the exosystem, along with a

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number of higher-order harmonics generated by the nonlinearities of the system. A linear servocompensator is then synthesized and augmented with the plant. Second, an output feedback controller is designed, using the separation approach of Esfandiari and Khalil [21], of first designing a state feedback controller, and then using a saturated high-gain observer to recover the performance of the state feedback design. And lastly, regional or semi-global stabilization of the augmented system, formed of the plant and the servocompensator, is achieved through a two-step approach, where robust control is first designed to guarantee convergence of the error to a neighbourhood of the zero-error manifold, followed by a locally high-gain control which stabilizes this disturbance-dependent manifold. By designing this robust, locally high-gain controller as a continuous sliding mode controller (CSMC), it is shown in Reference [17] that the only precise knowledge that is needed in this design is the relative degree of the plant, the sign of its high-frequency gain and the linear internal model.

In Reference [22], we presented a modification of the traditional integrator design in Reference [23] to address the issue of performance degradation caused by the inclusion of the integrator, specifically the issue of *integrator windup*. This was done by designing the integrator as a 'conditional' one, which provides integral action only inside the boundary layer. In this paper, we consider the counterpart of Seshagiri and Khalil [22] for the servomechanism design of Khalil [17], by designing the servocompensator as a conditional one. Both regional as well as semi-global asymptotic results are provided. We also analytically show that the controller can be tuned to recover the performance of an ideal (discontinuous) SMC. A result regarding ultimate boundedness of the tracking error under internal model perturbation is recalled [17], and a simulation example is included to demonstrate the improvement in transient performance over the 'conventional compensator' design of Khalil [17].

2. SYSTEM DESCRIPTION AND ASSUMPTIONS

Consider a single-input single-output nonlinear system, modelled by

$$\begin{aligned} \dot{x} &= f(x,\theta) + g(x,\theta)u + \beta(x,d,\theta) \\ y &= h(x,\theta) + \gamma(d,\theta) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R$ is the control input, $y \in R$ is the measured output, $d \in R^p$ is a time-varying disturbance input. The functions f, g, β, h and γ depend continuously on θ, a vector of unknown constant parameters, which belongs to a compact set $\Theta \subset R^l$. We assume that, for all $\theta \in \Theta$, the functions are sufficiently smooth on U_θ, an open connected subset of R^n that could depend on θ, for all d in a compact set of interest. The functions β and γ vanish at $d = 0$, i.e. $\beta(x, 0, \theta) = 0$ and $\gamma(0, \theta) = 0$ for all $\theta \in \Theta$ and $x \in U_\theta$. Our first assumption is that the disturbance-free system has a well-defined normal form, possibly with zero dynamics.

Assumption 1

System (1), with $d = 0$, has a uniform relative degree $\rho \leqslant n$ for all $x \in U_\theta$ and $\theta \in \Theta$; i.e. $L_g h(x,\theta) = L_g L_f h(x,\theta) = \cdots = L_g L_f^{\rho-2} h(x,\theta) = 0$ and $|L_g L_f^{\rho-1} h(x,\theta)| \geqslant g_0 > 0$ where g_0 is independent of θ. Moreover, there exists a diffeomorphism

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x,\theta) \tag{2}$$

of U_θ onto its image that transforms (1), with $d = 0$, into the normal form[§]

$$\begin{aligned} \dot{\eta} &= \phi(\eta, \xi, \theta) \\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \leqslant i \leqslant \rho - 1 \\ \dot{\xi}_\rho &= b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u \\ y &= \xi_1 \end{aligned} \tag{3}$$

Assumption 2

In the presence of disturbance, the change of variables (2) transforms the system into the form[¶]

$$\begin{aligned} \dot{\eta} &= \phi_a(\eta, \xi_1, \ldots, \xi_{m+1}, d, \theta) \\ \dot{\xi}_i &= \xi_{i+1} + \Psi_i(\xi_1, \ldots, \xi_i, d, \theta), \quad 1 \leqslant i \leqslant m - 1 \\ \dot{\xi}_i &= \xi_{i+1} + \Psi_i(\eta, \xi_1, \ldots, \xi_i, d, \theta), \quad m \leqslant i \leqslant \rho - 1 \\ \dot{\xi}_\rho &= b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u + \Psi_\rho(\eta, \xi, d, \theta) \\ y &= \xi_1 + \gamma(d, \theta) \end{aligned} \tag{4}$$

where $1 \leqslant m \leqslant \rho - 1$. The functions Ψ_i vanish at $d = 0$.

Examples of physical systems which are transformable into the normal form in Assumption 1, uniformly in a compact set of system parameters, can be found, for example in Reference [24, Section 4.10]. Geometric conditions under which a system can be transformed into the form in Assumption 2 can be found, for example, in Reference [25].

Assumption 3

Let ρ_0 be the disturbance relative degree and $\tilde{\rho} = \rho - \rho_0$. The disturbance and reference signals $d(t)$ and $r(t)$ have the following properties for all $t \geqslant 0$:

(i) $d(t)$ and its derivatives up to the $\tilde{\rho}$th derivative are bounded, and $d^{(\tilde{\rho})}(t)$ is piecewise continuous;

(ii) $r(t)$ and its derivatives up to the ρth derivative are bounded, and $r^{(\rho)}(t)$ is piecewise continuous;

(iii) $\lim_{t\to\infty} [\mathscr{D}(t) - \bar{\mathscr{D}}(t)] = 0$ and $\lim_{t\to\infty} [\mathscr{Y}(t) - \bar{\mathscr{Y}}(t)] = 0$, where $\mathscr{D}^{\mathrm{T}}(t) = [d(t) \cdots d^{(\tilde{\rho})}(t)]$, $\mathscr{Y}^{\mathrm{T}}(t) = [r(t) \cdots r^{(\rho)}(t)]$, and $\bar{\mathscr{D}}(t)$ and $\bar{\mathscr{Y}}(t)$ are generated by the known exosystem

$$\dot{w} = S_0 w$$

$$\begin{bmatrix} \bar{\mathscr{D}} \\ \bar{\mathscr{Y}} \end{bmatrix} = \Gamma_0 w \tag{5}$$

where S_0 has distinct eigenvalues on the imaginary axis and $w(t)$ belongs to a compact set W.

[§]For $\rho = n$, η and the $\dot{\eta}$-equation are dropped.

[¶]For $m = 1$, the first $\dot{\xi}_i$-equation is dropped.

Let D and Y be compact subsets of $R^{(\tilde{\rho}+1)p}$ and $R^{(\rho+1)}$, respectively, such that $\mathscr{D} \in D$ and $\mathscr{Y} \in Y$, and $\bar{d}(w)$ and $\bar{r}(w)$ denote the steady-state values of d and r as determined by exosystem (5). Define $\pi_1(w,\theta)$ to $\pi_m(w,\theta)$ by

$$\pi_1 = \bar{r} - \gamma(\bar{d},\theta)$$

$$\pi_{i+1} = \frac{\partial \pi_i}{\partial w} S_0 w - \Psi_i(\pi_1,\ldots,\pi_i,\bar{d},\theta), \quad 1 \leqslant i \leqslant m-1$$

Assumption 4
There exists a unique mapping $\lambda(w,\theta)$ that solves the partial differential equation

$$\frac{\partial \lambda}{\partial w} S_0 w = \phi_a(\lambda,\pi_1,\ldots,\pi_m,\pi_{m+1},\bar{d},\theta)$$

for all $w \in W$, where

$$\pi_{m+1} = \frac{\partial \pi_m}{\partial w} S_0 w - \Psi_m(\lambda,\pi_1,\ldots,\pi_m,\bar{d},\theta)$$

Let

$$\pi_{i+1} = \frac{\partial \pi_i}{\partial w} S_0 w - \Psi_i(\lambda,\pi_1,\ldots,\pi_i,\bar{d},\theta), \quad m+1 \leqslant i \leqslant \rho-1$$

The steady-state value of the control u on the zero-error manifold $\{\eta = \lambda(w,\theta), \xi = \pi(w,\theta)\}$ is given by

$$\chi(w,\theta) = \frac{1}{a(\lambda,\pi,\theta)}\left[\frac{\partial \pi_\rho}{\partial w} S_0 w - b(\lambda,\pi,\theta) - \Psi_p(\lambda,\pi,\bar{d},\theta)\right]$$

Assumption 5
There exists a set of real numbers $c_0,\ldots,c_{q-1}$, independent of θ, such that $\chi(w,\theta)$ satisfies the identity

$$L_s^q \chi = c_0 \chi + c_1 L_s \chi + \cdots + c_{q-1} L_s^{q-1} \chi \tag{6}$$

for all $(w,\theta) \in W \times \Theta$, where $L_s \chi = (\partial \chi/\partial w) S_0 w$ and the characteristic polynomial

$$p^q - c_{q-1} p^{q-1} - \cdots - c_0$$

has distinct roots on the imaginary axis.

Motivation for Assumption 5 comes from the nonlinear version of the internal model principle, which recognizes that in the nonlinear case, the controller must be able to reproduce not only the trajectories generated by the exosystem, but also a number of higher order nonlinear deformations thereof, an idea that was elaborated independently in References [7–9]. Assumption 5, along with the notion of *immersion* [24, Chapter 8], allows the construction of a finite-dimensional linear internal model, as we will soon show. However, before we do so, a couple of remarks are in order. Note that, among other things, the matrix S_0 in Assumption 3, and hence the frequencies of the exosystem need to be precisely known. For the case where the frequencies of the exosystem are unknown, an alternate design, that makes use of an adaptive internal model whose 'natural frequencies' are automatically tuned to match those of the unknown exosystem, can be found in Reference [26]. A recent result, which relaxes Assumption 5, thereby removing the restriction that the solution of the regulator equations be a polynomial in the exogenous signals, and allows for a nonlinear internal model, can be found in Reference [27].

Defining

$$S = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ c_0 & \cdots & \cdots & \cdots & c_{q-1} \end{bmatrix}, \quad \tau = \begin{bmatrix} \chi \\ L_s\chi \\ \vdots \\ L_s^{q-2}\chi \\ L_s^{q-1}\chi \end{bmatrix}$$

and $\Gamma = [1 \;\; 0 \;\; \cdots \;\; 0]_{1\times q}$, it can be shown that $\chi(w,\theta)$ is generated by the internal model

$$\frac{\partial \tau(w,\theta)}{\partial w} S_0 w = S\tau(w,\theta)$$

$$\chi(w,\theta) = \Gamma\tau(w,\theta)$$

To tackle the tracking problem, we apply the change of variables $z = \eta - \lambda(w,\theta)$ and $e_i = y^{(i-1)} - r^{(i-1)}$, $1 \leqslant i \leqslant \rho$ and $v^{\mathrm{T}}(t) = [\mathscr{D}^{\mathrm{T}}(t) - \bar{\mathscr{D}}^{\mathrm{T}}(t), \mathscr{Y}^{\mathrm{T}}(t) - \bar{\mathscr{Y}}^{\mathrm{T}}(t)]$, and note that $v(t)$ belongs to a compact set Λ and $\lim_{t\to\infty} v(t) = 0$. With this change of variables, system (4) can be rewritten as

$$\begin{aligned} \dot{z} &= \phi_0(z,e,v,w,\theta) \\ \dot{e}_i &= e_{i+1}, 1 \leqslant i \leqslant \rho - 1 \\ \dot{e}_\rho &= b_0(z,e,v,w,\theta) + a_0(z,e,v,w,\theta)u \\ y_m &= e_1 \end{aligned} \tag{7}$$

where y_m is the measured tracking error. The functions $\phi_0(\cdot), a_0(\cdot)$ and $b_0(\cdot)$ satisfy

$$\phi_0(0,0,0,w,\theta) = 0$$

$$a_0(0,0,0,w,\theta) = a(\lambda(w,\theta), \pi(w,\theta), \theta)$$

$$b_0(0,0,0,w,\theta) = -\chi(w,\theta)a(\lambda(w,\theta), \pi(w,\theta), \theta)$$

In the new variables, the zero-error manifold is given by $\{z = 0, e = 0\}$. Since we do not necessarily require our assumptions to hold globally, we need to restrict our analysis in the (z,e) variables to a region that maps back into the domain U_θ. The following assumption states such a restriction.

Assumption 6

There exist positive constants r_1 and r_2, independent of (v,w,θ), such that for all $(v,w,\theta) \in \Lambda \times W \times \Theta$, $\|e\| < r_1$ and $\|z\| < r_2 \Rightarrow x \in U_\theta$.

Define the balls $\mathscr{E} = \{e \in R^{\rho} : \|e\| < r_1\}$ and $\mathscr{Z} = \{z \in R^{n-\rho} : |z\| < r_2\}$. Since Λ is compact, there exists $r_3 > 0$ such that $\|v\| < r_3$ for all $v \in \Lambda$. Therefore, $\|(e^{\mathrm{T}}, v^{\mathrm{T}})\| < r_1 + r_3$ for all $e \in \mathscr{E}$ and $v \in \Lambda$. We will design the control u to regulate the error e to zero and then rely on a minimum-phase-like assumption, stated below, to guarantee boundedness of z. The assumption states that with $(e^{\mathrm{T}}, v^{\mathrm{T}})$ as the driving input, the system $\dot{z} = \phi_0(z, e, v, w, \theta)$ is input-to-state stable over a certain region [28, Theorem 4.19], which implies that with $(e^{\mathrm{T}}, v^{\mathrm{T}}) = 0$, the equilibrium point $z = 0$ is asymptotically stable. This is strengthened in Assumption 8 to requiring that this equilibrium point be locally exponentially stable.

Assumption 7
There exist a C^1 proper function $V_z : \mathscr{Z} \times W \to R_+$, possibly dependent on θ, and class $\mathscr{K}$ functions $\alpha_i : [0, r_2) \to R_+ (i = 1, 2, 3)$ and $\delta : [0, r_1 + r_3) \to R_+$, independent of (w, θ), such that

$$\alpha_1(\|z\|) \leqslant V_z(z, w, \theta) \leqslant \alpha_2(\|z\|)$$

$$\frac{\partial V_z}{\partial z} \phi_0(z, e, v, w, \theta) + \frac{\partial V_z}{\partial w} S_0 w \leqslant -\alpha_3(\|z\|)$$

for all $\|z\| \geqslant \delta(\|(e^{\mathrm{T}}, v^{\mathrm{T}})\|), \|(e^{\mathrm{T}}, v^{\mathrm{T}})\| < r_1 + r_3$ and $(z, w, \theta) \in \mathscr{Z} \times W \times \Theta$. Furthermore, $\delta(r_3) < \alpha_2^{-1}(\alpha_1(r_2))$.[‖]

Assumption 8
There exists a Lyapunov function $V_{zz}(z, w, \theta)$, defined in some neighbourhood of $z = 0$, and positive constants λ_1 to λ_4, independent of (w, θ), such that

$$\lambda_1 \|z\|^2 \leqslant V_{zz}(z, w, \theta) \leqslant \lambda_2 \|z\|^2$$

$$\frac{\partial V_{zz}}{\partial z} \phi_0(z, 0, 0, w, \theta) + \frac{\partial V_{zz}}{\partial w} S_0 w \leqslant -\lambda_3 \|z\|^2$$

$$\left\| \frac{\partial V_{zz}}{\partial z} \right\| \leqslant \lambda_4 \|z\|$$

3. CONTROL DESIGN

Our design of the conditional servocompensator follows very close to that of the conditional integrator in Reference [22]. Basically, it involves modifying the servocompensator

$$\dot{\sigma} = S\sigma + Je_1, \quad J^{\mathrm{T}} = [0, \ldots, 0, 1]$$

in Reference [17] to make it 'active' only inside the boundary layer. Assume for the present that the state e is available for feedback. To simplify the notation in what is to come, we define $\zeta^{\mathrm{T}} = [e_1 \;\; e_2 \;\; \cdots \;\; e_{\rho-1}]$ and $K_2 = [k_1 \;\; k_2 \;\; \cdots \;\; k_{\rho-1}]$. In the absence of the servocompensator, one

‖ ISS analysis on a finite region requires (see Reference [28, Theorem 4.18]) $\delta(\sup_{t \geqslant 0} \|(e^{\mathrm{T}}(t), v^{\mathrm{T}}(t))\|) < \alpha_2^{-1}(\alpha_1(r_2))$. Since $\|v\| < r_3$, Assumption 7 requires that $\delta(r_3) < \alpha_2^{-1}(\alpha_1(r_2))$. Later on in the analysis, a restriction is placed on $\|e\|$ (see (16)).

could take the sliding surface as $s = K_2\zeta + e_p$, with K_2 chosen such that the polynomial $\lambda^{\rho-1} + k_{p-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$ is Hurwitz. This guarantees that when motion is confined to the manifold $s = 0$, the error e_1 converges to zero asymptotically. Servocompensation is then introduced by modifying the sliding surface to

$$s = K_1\sigma + K_2\zeta + e_\rho \tag{8}$$

where σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J \,\mathrm{sat}(s/\mu) \tag{9}$$

with $\mu > 0$ being the width of the boundary layer and K_1 chosen such that $S - JK_1$ is Hurwitz, which is always possible since the pair (S, J) is controllable. Equation (9) represents a perturbation of the exponentially stable system $\dot{\sigma} = (S - JK_1)\sigma$, with the norm of the perturbation bounded by the small parameter μ. Inside the boundary layer, i.e. when $|s| \leqslant \mu$, Equation (9) reduces to

$$\dot{\sigma} = S\sigma + Je_{\mathrm{a}} \tag{10}$$

where the 'augumented error' $e_{\mathrm{a}} = K_2\zeta + e_\rho$ is a linear combination of the tracking error and its derivatives up to order $\rho - 1$. Equation (10) coincides with the servocompensator of Khalil [17] only in the case when $\rho = 1$.

Since the state e is unavailable for feedback, we use the following linear high-gain observer to robustly estimate the derivatives of e_1:

$$\begin{aligned} \dot{\hat{e}}_i &= \hat{e}_{i+1} + g_i(e_1 - \hat{e}_1)/\varepsilon^i, \quad 1 \leqslant i \leqslant \rho - 1 \\ \dot{\hat{e}}_\rho &= g_\rho(e_1 - \hat{e}_1)/\varepsilon^\rho \end{aligned} \tag{11}$$

where $\varepsilon > 0$ is a design parameter to be specified, and the positive constants $g_1, \ldots, g_\rho$ are chosen such that the polynomial $\lambda^\rho + g_1\lambda^{\rho-1} + \cdots + g_{\rho-1}\lambda + g_\rho$ is Hurwitz. We replace s by its estimate $\hat{s}$, given by

$$\hat{s} = K_1\sigma + k_1 e_1 + \sum_{i=2}^{\rho-1} k_i \hat{e}_i + \hat{e}_\rho \tag{12}$$

where σ is the output of

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J \,\mathrm{sat}(\hat{s}/\mu) \tag{13}$$

The control is taken as

$$u = -k \,\mathrm{sign}(L_g L_f^{\rho-1} h) \,\mathrm{sat}(\hat{s}/\mu) \tag{14}$$

To complete the design, we need to specify the design parameters k, μ and ε. The constant k is an upper bound on the control u. Since in typical applications the control has to satisfy magnitude constraints, we can simply choose k to be the maximum permissible control magnitude. The parameters μ and ε should be chosen sufficiently small. In particular, we show in the next section that there exists $\mu^* > 0$, and for each $\mu \in (0, \mu^*)$, there is an $\varepsilon^* = \varepsilon^*(\mu) > 0$ such that asymptotic tracking is achieved for each $0 < \mu < \mu^*$ and $0 < \varepsilon < \varepsilon^*$. The only precise knowledge about the plant that is required to calculate and implement control (14) is its relative degree ρ, the sign of its high-frequency gain $L_g L_f^{\rho-1} h$, and the characteristic polynomial of Assumption 5.

4. ANALYSIS

We write the closed-loop system in the singularly perturbed form

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{z} &= \phi_0(z, e, v, w, \theta) \\
\dot{\sigma} &= A_\sigma \sigma + \mu J \,\mathrm{sat}((s - N(\varepsilon)\varphi)/\mu) \\
\dot{\zeta} &= A_\zeta \zeta + B_1(s - K_1\sigma) \\
\dot{s} &= \Delta(\cdot) - k|a_0(\cdot)| \,\mathrm{sat}((s - N(\varepsilon)\varphi)/\mu) \\
\varepsilon\dot{\varphi} &= A_\varphi \varphi + \varepsilon B_2[b_0(\cdot) - k|a_0(\cdot)| \,\mathrm{sat}((s - N(\varepsilon)\varphi)/\mu)]
\end{aligned} \tag{15}$$

where

$$A_\zeta = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -k_1 & -k_2 & \dots & \dots & -k_{\rho-1} \end{bmatrix}$$

$$A_\varphi = \begin{bmatrix} -g_1 & 1 & \dots & \dots & 0 \\ \vdots & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ \vdots & \dots & \dots & 0 & 1 \\ -g_\rho & 0 & \dots & \dots & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\Delta(\cdot) = b_0(\cdot) + K_1 A_\sigma \sigma + \mu K_1 J \,\mathrm{sat}((s - N(\varepsilon)\varphi)/\mu) + K_2[e_2 \;\; e_3 \;\; \cdots \;\; e_\rho]^{\mathrm{T}}$$

$$N(\varepsilon) = [0 \;\; k_2\varepsilon^{\rho-2} \;\; k_3\varepsilon^{\rho-3} \;\; \cdots \;\; k_{\rho-1}\varepsilon \;\; 1]$$

$A_\sigma \overset{\text{def}}{=} S - JK_1$, and the scaled estimation φ is given by

$$\varphi_i = \frac{1}{\varepsilon^{\rho-i}}(e_i - \hat{e}_i), \quad 1 \leqslant i \leqslant \rho$$

The current analysis shares many points in common with the ones in References [17, 22], so we only outline the idea, taking care to highlight the differences. The main difference between the current analysis and that in Reference [17] is treating σ and ζ separately in the current work, while in Reference [17], they are lumped together in one vector. The four main steps in the analysis, common to Reference [17] and the current work, are as follows:

Step 1: Using appropriate Lyapunov functions for each of the last five components of (15), define a compact set $\Omega_c \times \Sigma_\varepsilon$ and show that this set is a positively invariant set of the closed-loop system for a suitable choice of controller parameters. This is done by showing that the derivative of the Lyapunov functions are negative on the Lyapunov surfaces that form the boundaries of this set.

Step 2: Show that for any bounded $\hat{e}(0)$, and any $(z(0), e(0), \sigma(0)) \in \Omega_b$, where $0 < b < c$, it is possible to choose ε such that the trajectory enters the set $\Omega_c \times \Sigma_\varepsilon$ in finite time.

Step 3: Show that for a suitable choice of μ and ε, all trajectories starting inside $\Omega_c \times \Sigma_\varepsilon$ eventually enter into a 'small' positively invariant set $\Psi_{\mu,\varepsilon}$ that shrinks to the origin as μ and ε tend to zero.

Step 4: Show that for a suitable choice of μ and ε, every trajectory in $\Psi_{\mu,\varepsilon}$ approaches an invariant manifold on which the error is zero.

Noting that A_ζ, A_φ, and A_σ are Hurwitz, we define the Lyapunov functions

$$V_\zeta(\zeta) \stackrel{\text{def}}{=} \zeta^{\mathrm{T}} P_\zeta \zeta, \quad V_\varphi(\varphi) \stackrel{\text{def}}{=} \varphi^{\mathrm{T}} P_\varphi \varphi \quad \text{and} \quad V_\sigma(\sigma) \stackrel{\text{def}}{=} \sigma^{\mathrm{T}} P_\sigma \sigma$$

where the symmetric positive definite matrices P_ζ, P_φ, and P_σ are the solutions of

$$P_\zeta A_\zeta + A_\zeta^{\mathrm{T}} P_\zeta = -I$$

$$P_\varphi A_\varphi + A_\varphi^{\mathrm{T}} P_\varphi = -I$$

and

$$P_\sigma A_\sigma + A_\sigma^{\mathrm{T}} P_\sigma = -I$$

respectively. Given a positive constant $c > \mu$, define the compact set Ω_c by

$$\Omega_c \stackrel{\text{def}}{=} \{(z, e, \sigma) : |s| \leqslant c, \ V_\sigma(\sigma) \leqslant \mu^2 \rho_1, \ V_\zeta(\zeta) \leqslant (c + \mu\rho_2)^2 \rho_3, \ V_z(t, z, d) \leqslant \alpha_4(c\rho_4 + r_3)\}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are positive constants, independent of c, to be specified shortly and α_4 is a class $\mathscr{K}$ function defined in terms of the functions α_2 and δ of Assumption 7 by $\alpha_4 = \alpha_2 \circ \delta$. Our analysis will be restricted to trajectories starting inside a product set of which Ω_c is a component. Therefore, in view of Assumption 6, Ω_c will have to be chosen to ensure that $(z, e, \sigma) \in \Omega_c$ implies that $(z, e) \in \mathscr{Z} \times \mathscr{E}$. Using

$$e = K_3\zeta + B_2(s - K_1\sigma), \quad K_3 = \begin{bmatrix} I \\ -K_2 \end{bmatrix}$$

it can be verified that inside Ω_c, $\|e\| \leqslant \rho_4 c$, where $\rho_4 = (1 + \rho_2)\|K_3\|\sqrt{(\rho_3/\lambda_{\min}(P_\zeta)} + 1 + \|K_1\|\sqrt{(\rho_1/\lambda_{\min}(P_\sigma)}$. Using this, along with Assumption 7, it can be shown that choosing c to satisfy

$$c\rho_4 < \min\{r_1, \alpha_4^{-1}(\alpha_1(r_2)) - r_3\} \tag{16}$$

guarantees that $(z, e) \in \mathscr{Z} \times \mathscr{E}$ for all $(z, e, \sigma) \in \Omega_c$.

Define the compact set Σ_ε by

$$\Sigma_\varepsilon \stackrel{\text{def}}{=} \{\varphi : V_\varphi(\varphi) \leqslant \varepsilon^2 \rho_5\}$$

where ρ_5 is a positive constant to be specified shortly. We wish to show that for a suitable choice of the controller parameters, the set $\Omega_c \times \Sigma_\varepsilon$ is a positively invariant set of the closed-loop system (15). Using the inequality

$$\dot{V}_\sigma \leqslant -\|\sigma\|^2 + 2\mu\|\sigma\|\|P_\sigma J\|$$

it is easy to show that $\dot{V}_\sigma \leqslant 0$ on the boundary $V_\sigma = \mu^2\rho_1$ for the choice $\rho_1 = 4\|P_\sigma J\|^2 \lambda_{\max}(P_\sigma)$. Inside the set Ω_c, $\|\sigma\| \leqslant \mu\sqrt{\rho_1/\lambda_{\min}(P_\sigma)} \stackrel{\text{def}}{=} \mu\rho_2/\|K_1\|$. Using this, along with $|s| \leqslant c$, and the inequality

$$\dot{V}_\zeta \leqslant -\|\zeta\|^2 + 2\|\zeta\|\ \|P_\zeta B_1\|\ (|s| + \|K_1\|\|\sigma\|)$$

it is easy to show that $\dot{V}_\zeta \leqslant 0$ on the boundary $V_\zeta = (c + \mu\rho_2)^2\rho_3$, for the choice $\rho_3 = 4\|P_\zeta B_1\|^2 \lambda_{\max}(P_\zeta)$. Next we evaluate $s\dot{s}$ on the boundary $|s| = c$. To that end, let ε be small enough that $|N(\varepsilon)\varphi| \leqslant c - \mu$, so that

$$\text{sat}\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right) = \text{sgn}\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right) = \text{sgn}(s)$$

and hence

$$s\dot{s} \leqslant |\Delta(\cdot)|\ |s| - k|a_0(\cdot)|\ |s|$$

Choosing k and c to satisfy**

$$k \geqslant \rho_6 + \gamma_1(c) \tag{17}$$

where $\rho_6 > 0$ and $\gamma_1(c) = \max|\Delta(\cdot)|/|a_0(\cdot)|$, with the maximization taken over all $(z, e, \sigma) \in \Omega_c$, $v \in \Lambda$, $w \in W$, and $\theta \in \Theta$, we have $s\dot{s} < 0$ on the boundary $|s| = c$. Assumption 7 shows that $\dot{V}_z \leqslant 0$ on the boundary $V_z = \alpha_4(c\rho_4 + r_3)$. Finally, using the inequality

$$\dot{V}_\varphi \leqslant -\frac{1}{\varepsilon}\|\varphi\|^2 + 2\|\varphi\|\ \|P_\varphi B_2\|\gamma_2(c)$$

where $\gamma_2(c) = \max|b_0(\cdot) - k|a_0(\cdot)|\text{sat}((s - N(\varepsilon)\varphi)/\mu)|$, with the maximization taken over the same set as that for $\gamma_1(c)$, it follows that $\dot{V}_\varphi \leqslant 0$ on the boundary $V_\varphi = \varepsilon^2\rho_5$ for the choice $\rho_5 > 4\|P_\varphi B_2\|^2\gamma_2^2(c)\lambda_{\max}(P_\varphi)$. It follows that the set $\Omega_c \times \Sigma_\varepsilon$ is positively invariant.

Our next step is to show that for any bounded $\hat{e}(0)$, and any $(z(0), e(0), \sigma(0)) \in \Omega_b$, where $0 < b < c$, it is possible to choose ε such that the trajectory enters the set $\Omega_c \times \Sigma_\varepsilon$ in finite time. Using the fact that for $(z, e, \sigma) \in \Omega_c$, the right-hand side of the slow equation of (15) is bounded uniformly in ε, if follows that for all $(z(0), e(0), \sigma(0)) \in \Omega_b$, there is a finite time T_0, independent of ε such that for all $0 \leqslant t \leqslant T_0$, $(z(t), e(t), \sigma(t)) \in \Omega_c$. During this interval, using the definition of ρ_5, we have

$$\dot{V}_\varphi \leqslant -\rho_7\varepsilon\|\varphi\|^2 \quad \text{for } V_\varphi(\varphi) \geqslant \varepsilon^2\rho_5$$

for some $\rho_7 > 0$. This inequality can be used to show that $\varphi(t)$ enters Σ_ε within a time interval $[0, T(\varepsilon)]$, where $\lim_{\varepsilon\to 0} T(\varepsilon) = 0$. Therefore, by choosing ε small enough we can ensure that $T(\varepsilon) < T_0$.

** Inequality (17) can be viewed in two ways. Given $c > 0$, it is a constraint on the minimum value k. Alternatively, given k, it is a constraint on the estimate of the region of attraction.

The argument that $\Omega_c \times \Sigma_\varepsilon$ is positively invariant can be extended to show that, for sufficiently small ε, all trajectories starting inside it reach the positively invariant set $\Psi_{\mu,\varepsilon} \stackrel{\text{def}}{=} \Omega_\mu \times \Sigma_\varepsilon$ in finite time, where

$$\Omega_\mu \stackrel{\text{def}}{=} \{(z,e,\sigma) : |s| \leqslant \mu(1-\delta_0), V_\sigma(\sigma) \leqslant \mu^2 \rho_1,$$
$$V_\zeta(\zeta) \leqslant \mu^2 \rho_8, V_z(t,z,d) \leqslant \alpha_4(\mu\rho_9)\}$$

where $0 < \delta_0 < \rho_6/(4k) < 1/4$, ε is small enough that $|N(\varepsilon)\varphi| < \mu\delta_0$, and ρ_8, ρ_9 are positive constants independent of μ. The details of showing that the trajectories reach $\Psi_{\mu,\varepsilon}$ in finite time can be found in Reference [29]. The idea can be explained roughly as follows; note that Ω_μ has four components; (i) the standard sliding mode analysis tells us that the component $|s|$ becomes $O(\mu)$ in finite time, (ii) from our previous analysis $\|\sigma\|$ is always $O(\mu)$, (iii) using (i) and (ii) along with the $\dot{\zeta}$ equation of (15), it can be shown that $\|\zeta\|$ becomes $O(\mu)$, and (iv) lastly, using (i), (ii) and (iii) along with the fact that $\lim_{t\to\infty} v(t) = 0$ and Assumption 7, it can be shown that $\|z\|$ becomes $O(\mu)$. We previously showed that $\|\varphi\|$ is $O(\varepsilon)$. It is clear that the set $\Psi_{\mu,\varepsilon}$ shrinks to the origin as $\mu, \varepsilon \to 0$.

Lastly, we show that every trajectory in $\Psi_{\mu,\varepsilon}$ approaches an invariant manifold on which the error is zero. To do this, we first note that inside $\Psi_{\mu,\varepsilon}$, the closed-loop system is given by

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{z} &= \phi_0(z,e,v,w,\theta) \\
\dot{\sigma} &= A_\sigma \sigma + J(s - N(\varepsilon)\varphi) \\
\dot{\zeta} &= A_\zeta \zeta + B_1(s - K_1\sigma) \\
\dot{s} &= \Delta(\cdot) - k|a_0(\cdot)|\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right) \\
\varepsilon\dot{\varphi} &= A_\varphi \varphi + \varepsilon B_2\left[b_0(\cdot) - k|a_0(\cdot)|\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right)\right]
\end{aligned} \tag{18}$$

Next, we claim that there exists a unique matrix M such that

$$SM = MS \quad \text{and} \quad -K_1 M = \Gamma$$

To see this, note that since A_σ is Hurwitz and S has eigenvalues on the imaginary axis, the Sylvester equation $A_\sigma X - XS = J\Gamma$ has a unique solution. That this solution satisfies $SX - XS = 0$ and $K_1 X + \Gamma = 0$ is shown in Reference [30]. Thus $M = X$ is the required matrix. Defining

$$\mathscr{M}_\mu = \{z = 0, \sigma = \bar{\sigma}, e = 0, \varphi = 0\}$$

where

$$\bar{\sigma} = (\mu/k)\ \text{sign}(L_g L_f^{\rho-1} h) M_\tau(w,\theta)$$

it is easy to verify by direct substitution that $\mathscr{M}_\mu$ is an invariant manifold of (18) when $v = 0$, for all $w \in W$. Consider the Lyapunov function candidate

$$V = V_{zz}(z,w,\theta) + \lambda_5 V_\zeta(\zeta) + \lambda_6 V_\sigma(\tilde{\sigma}) + \tfrac{1}{2}\tilde{s}^2 + V_\varphi(\varphi) \tag{19}$$

where $\tilde{\sigma} = \sigma - \bar{\sigma}$, $\tilde{s} = s - \bar{s}$, $\bar{s} = K_1\bar{\sigma}$, and λ_5, λ_6 are positive constants. By considering each term in the derivative of V of (19) separately, it can be shown [29] that the derivative of V can be arranged in the following quadratic form of $\Pi = [\|z\| \ \|\zeta\| \ \|\tilde{\sigma}\| \ |\tilde{s}| \ \|\varphi\|]^{\mathrm{T}}$

$$\dot{V} \leqslant -\Pi^{\mathrm{T}}\mathscr{P}\Pi + \lambda_7\|\Pi\|\,\|v\| \tag{20}$$

where λ_7 is a positive constant, and the symmetric matrix $\mathscr{P}$ has the form

$$\mathscr{P} = \begin{bmatrix} \lambda_3 & -\lambda_{1a} & -\lambda_{1b} & -\lambda_{1c} & -\lambda_{1d} \\ & \lambda_5 & -\lambda_{2b} & -\lambda_{2c} & -\lambda_{2d} \\ & & \lambda_6 & -\lambda_{3c} & -\lambda_{3d} \\ & & & \dfrac{kg_0}{\mu} - \lambda_{4c} & -\lambda_{4d} \\ & & & & \dfrac{1}{\varepsilon} - \lambda_{5d} \end{bmatrix}$$

where the nonnegative constants λ_{1d} to λ_{5d} are independent of ε; λ_{1c} to λ_{4c} are independent of μ; λ_{1b} and λ_{2b} are independent of λ_6; and λ_{1a} is independent of λ_5. By choosing λ_5, λ_6, μ, and ε, we can successively make the principal leading minors of $\mathscr{P}$ positive. First λ_5 is chosen large enough to make the 2×2 minor positive, then λ_6 is chosen large enough to make the 3×3 minor positive, next μ is chosen small enough to make the 4×4 minor positive, finally ε is chosen small enough to make $\mathscr{P}$ positive definite. Since $v(t) \to 0$ as $t \to \infty$, (20) can then be used to show that every trajectory inside $\Psi_{\mu,\varepsilon}$ approaches $\mathscr{M}_\mu$ as $t \to \infty$. Our conclusions can be summarized in the following theorem.

Theorem 1

Suppose Assumptions 1–8 are satisfied and consider the closed-loop system formed of plant (7) and the output feedback controller (11)–(14). Suppose $\hat{e}(0)$ belongs to a compact set $Q \in R^\rho$ and the initial states $(z(0), \sigma(0), e(0)) \in \Omega_b$, where $0 < b < c$, and c satisfies (16) and (17). Then, there exists $\mu^* > 0$, dependent on c, and for each $\mu \in (0, \mu^*]$, there exists $\varepsilon^* = \varepsilon^*(\mu)$, dependent on Q, such that for all $\mu \in (0, \mu^*]$ and $\varepsilon \in (0, \varepsilon^*]$, all the state variables of the closed-loop system are bounded and $\lim_{t\to\infty} e(t) = 0$.

The estimate of the region of attraction Ω_b is limited only by two factors: the region of validity of our assumptions, and the control level k. If all the assumptions hold globally and k can be chosen arbitrarily large, the controller can achieve semi-global regulation.

We conclude this section with the following theorem on the performance of the controller, which states that the controller recovers the performance of ideal state-feedback SMC, without servocompensation. Consider the ideal SMC

$$\begin{aligned} u &= -k\,\mathrm{sign}(L_g L_f^{\rho-1} h)\mathrm{sgn}(s) \\ s &= \sum_{i=2}^{\rho-1} k_i e_i + e_\rho \end{aligned} \tag{21}$$

Theorem 2

Let $X = (z, e)$ be part of the state of the closed-loop system for system (7) with the output feedback control (11)–(14) and $X^* = (z^*, e^*)$ be the state of the closed-loop system with the state feedback control (21), with $X(0) = X^*(0)$. Then, under the hypotheses of Theorem 1, for every $\varrho > 0$, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*]$, there exists $\varepsilon^* = \varepsilon^*(\mu)$, such that for all $\mu \in (0, \mu^*]$ and $\varepsilon \in (0, \varepsilon^*]$, $\|X(t) - X^*(t)\| \leqslant \varrho \ \forall t \geqslant 0$.

Proof

We prove the theorem in two parts. First, we compare the trajectories under ideal SMC with those under state feedback continuous SMC with the conditional servocompensator. Let $X^\dagger = (z^\dagger, e^\dagger)$ be part of the state of the closed-loop system under the control (8), (9) and

$$u = -k\,\mathrm{sign}(L_g L_f^{\rho-1} h)\mathrm{sat}(s/\mu)$$

with $X^\dagger(0) = X^*(0)$. For this case, we show that, for sufficiently small μ, $X^\dagger(t) - X^*(t) = O(\mu)\ \forall t \geqslant 0$. Let $s^\dagger$ and s^* be the corresponding sliding surface functions of the two systems and

$$t_0 = \min\{t^* : |s^\dagger(t)| \leqslant \mu(1 + \rho_2)\ \forall t \geqslant t^*\}$$

If $t_0 > 0$, then since $|K_1\sigma^\dagger(t)| \leqslant \mu\rho_2\ \forall t$, it follows that

$$\mathrm{sat}(s^\dagger(t)/\mu) = \mathrm{sgn}(s^\dagger(t)) = \mathrm{sgn}(s^\dagger(t) - K_1\sigma^\dagger(t))$$

$\forall\ 0 \leqslant t \leqslant t_0$, which can then be used to show that $X^\dagger(t) = X^*(t)\ \forall\ 0 \leqslant t < t_0$. The result holds trivially if $t_0 = 0$. We now consider $X^\dagger(t)$ and $X^*(t)$ in the time interval $t \geqslant t_0$. Since $X^\dagger(t_0) = X^*(t_0)$, we have $|s^\dagger(t_0) = s^*(t_0)| = |K_1\sigma^\dagger(t_0)| \leqslant \mu\rho_2$. Using this, along with the fact that $|s^\dagger(t)|$ and $|s^*(t)|$ monotonically converge to the positively invariant sets $\{|s^\dagger| \leqslant \mu\}$ and $\{0\}$, respectively, it can be shown that $|s^\dagger(t) - s^*(t)| \leqslant 2\mu(1 + \rho_2)$ for all $t \geqslant t_0$. It follows that $s^\dagger(t) - s^*(t) = O(\mu)\ \forall t \geqslant 0$. Since the equations for $\zeta^\dagger$ and ζ^* are identical stable linear equations, driven by inputs $s^\dagger - K_1\sigma^\dagger$ and s^*, respectively, where $|K_1\sigma^\dagger| \leqslant \mu\rho_2$ and $s^\dagger - s^* = O(\mu)$, continuity of solutions on the infinite time interval [28, Theorem 9.1] can be used to show that for sufficiently small μ, $\zeta^\dagger(t) - \zeta^*(t) = O(\mu)$ and hence $e^\dagger(t) - e^*(t) = O(\mu)$ for all $t \geqslant t_0$, which can then be used to show that $z^\dagger(t) - z^*(t) = O(\mu)$ for all $t \geqslant t_0$, so that the first part of the proof follows. In particular, there exists $\mu^* > 0$ such that

$$\mu \in (0, \mu^*] \Rightarrow \|X^\dagger(t) - X^*(t)\| \leqslant \tau/2 \quad \forall t \geqslant 0$$

In the second part of the proof, we use the idea in Reference [31] to show that the trajectories X of the system under output feedback approach the trajectories $X^\dagger$ under state feedback as $\varepsilon \to 0$. In particular, we show that there exists $\varepsilon^* = \varepsilon^*(\mu)$ such that for all $\varepsilon \leqslant \varepsilon^*$, $\|X(t) - X^\dagger(t)\| \leqslant \tau/2\ \forall t \geqslant 0$. This is done by dividing the time interval $[0, \infty)$ into three sub-intervals $[0, T(\varepsilon)], [T(\varepsilon), T_1]$ and $[T_1, \infty)$ and showing that the inequality $\|X(t) - X^\dagger(t)\| \leqslant \tau/2$ holds over each of these sub-intervals. From asymptotic stability of the two systems, we know that there exists a finite time T_1, independent of ε, such that $\|X(t) - X^\dagger(t)\| \leqslant \tau/2\ \forall t \geqslant T_1$. Also, as mentioned earlier in this section, there is a time interval $[0, T(\varepsilon)]$, with $T(\varepsilon) \to 0$ as $\varepsilon \to 0$, during which the fast variable φ decays to an $O(\varepsilon)$ value. It can be shown that global boundedness of the controls implies that over this interval, $\|X(t) - X^\dagger(t)\| \leqslant \lambda_0 T(\varepsilon)$, for some positive constant λ_0 that is independent of ε. Since $T(\varepsilon) \to 0$ as $\varepsilon \to 0$, for small enough ε,

$\|X(t)-X^{\dagger}(t)\|\leqslant\tau/2\ \forall t\in[0,T(\varepsilon)]$. Lastly, noting that $X(T(\varepsilon))-X^{\dagger}(T(\varepsilon))\to 0$ as $\varepsilon\to 0$ and φ is $O(\varepsilon)$, and using the continuous dependence of the solutions of differential equations on compact time intervals [28, Theorem 3.4], one can show that it is possible to choose ε to satisfy the inequality $\|X(t)-X^{\dagger}(t)\|\leqslant\tau/2$ over the time interval $[T(\varepsilon),T_1]$. In particular, there exists $\varepsilon^{*}=\varepsilon^{*}(\mu)>0$ such that

$$\varepsilon\in(0,\varepsilon^{*}]\Rightarrow\|X(t)-X^{\dagger}(t)\|\leqslant\tau/2\quad\forall t\geqslant 0$$

The conclusion of Theorem 2 then follows from the triangle inequality. □

5. INTERNAL MODEL PERTURBATION

As mentioned in the concluding remarks in Section 3, our design requires that the constants c_0 to c_q in Assumption 5, and hence the frequencies of the exosystem, be precisely known. In addition, as noted in the remarks following it, Assumption 5 is equivalent to requiring that the control input, when restricted to the zero-error manifold, be a polynomial function of the exogenous signals [14]. A violation of either of these conditions results in a perturbation of the internal model. To make the idea precise, let

$$\chi(w,\theta)=-b_0(0,0,0,w,\theta)/a_0(0,0,0,w,\theta)$$

and suppose $\bar{\chi}(w,\theta)$ is a nominal value of $\chi(w,\theta)$ that satisfies (6). As mentioned above, there are two sources for this perturbation. First, the frequencies of the exosystem are unknown, so that χ satisfies (6) with unknown coefficients c_0 to c_q, while $\bar{\chi}$ does so with nominal coefficients $\bar{c}_0$ to $\bar{c}_q$, which are used to construct the internal model. Second, the assumption that the control input, when restricted to the zero-error manifold, is a polynomial function of the exogenous signals, does not hold, so that χ does not satisfy (6), but an approximation $\bar{\chi}$ of it does. In this case, we note that since any continuous function can be approximated to arbitrary accuracy on compact sets by polynomials, a linear internal model that generates $\bar{\chi}$ can be used to approximate χ arbitrarily closely. Regardless of the source of the perturbation, using the results of Khalil [17, Section 5], it can be shown that provided the perturbation is small, the controller of the previous sections achieves ultimate boundedness of the tracking error, with the bound being proportional to the size of the perturbation. In particular, let $\tilde{\chi}(w,\theta)=\bar{\chi}(w,\theta)-\chi(w,\theta)$, and suppose

$$|\tilde{\chi}(w,\theta)\leqslant|\delta_{\chi}\ \forall(w,\theta)\in W\times\Theta$$

The analysis proceeds exactly as in Section 4 up to the point of showing that the trajectories enter the set $\Psi_{\mu,\varepsilon}$. Using the fact that

$$\begin{aligned}b_0(0,0,0,w,\theta)&=-\chi(w,\theta)a(\lambda,\pi,\theta)\\&=-\bar{\chi}(w,\theta)a(\lambda,\pi,\theta)+\tilde{\chi}(w,\theta)a(\lambda,\pi,\theta)\end{aligned}$$

it can be shown that inside the set $\Psi_{\mu,\varepsilon}$, when $v=0$, the closed-loop equation (18) can be written as

$$\begin{aligned}\dot{w}&=S_0w\\\dot{z}&=\phi_0(z,e,0,w,\theta)\\\dot{\sigma}&=A_\sigma\sigma+J(s-N(\varepsilon)\varphi)\end{aligned}\tag{22}$$

$$\dot{\zeta} = A_\zeta \zeta + B_1(s - K_1\sigma)$$

$$\dot{s} = \bar{\Delta}(z, \sigma, e, \varphi, 0, w, \theta) + \tilde{\chi}(w, \theta)a(\lambda, \pi, \theta) - k|a_0(z, \sigma, e, 0, w, \theta)|\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right)$$

$$\varepsilon\dot{\varphi} = A_\varphi \varphi + \varepsilon B_2\left[\bar{b}_0(z, \sigma, e, 0, w, \theta) - k|a_0(\cdot, 0, \cdot)|\left(\frac{s - N(\varepsilon)\varphi}{\mu}\right) + \tilde{\chi}(w, \theta)a(\lambda, \pi, \theta)\right]$$

where

$$\bar{\Delta}(z, \sigma, e, \varphi, v, w, \theta) = \Delta(z, \sigma, e, \varphi, v, w, \theta) - b_0(0, 0, 0, 0, w, \theta) - \tilde{\chi}(w, \theta)a(\lambda, \pi, \theta)$$

$$\bar{b}_0(z, \sigma, e, v, w, \theta) = b_0(z, \sigma, e, v, w, \theta) - b_0(0, 0, 0, 0, w, \theta) - \tilde{\chi}(w, \theta)a(\lambda, \pi, \theta)$$

It can be verified that system (22) has $\mathcal{M}_\mu$ as an invariant manifold when $\tilde{\chi} = 0$. Equation (22) takes the form of Reference [17, Equation (A1)], and satisfies all the assumptions of Reference [17, Lemma 2], so that the results of Reference [17, Lemma 2] can be applied to show that (22) has an exponentially attractive manifold $\bar{\mathcal{M}}_\mu$ that is $O(\delta_\chi)$ close to $\mathcal{M}_\mu$, and on which $e = O(\delta_\chi)$. The Lyapunov analysis of the final part of the proof of Theorem 1 can be repeated to show that all trajectories inside $\Psi_{\mu,\varepsilon}$ approach $\bar{\mathcal{M}}_\mu$ as t tends to infinity. Our results can be summarized in the following theorem.

Theorem 3

Under the hypotheses of Theorem 1, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*]$, there exists $\varepsilon^* = \varepsilon^*(\mu) > 0$ and $\delta_\chi^* = \delta_\chi^*(\mu) > 0$, such that for all $\delta_\chi \in [0, \delta_\chi^*]$, $\mu \in (0, \mu^*]$ and $\varepsilon \in (0, \varepsilon^*]$, all the state variables of the closed-loop system are bounded and converge to an invariant manifold where $e = O(\delta_\chi)$.

6. SIMULATION EXAMPLE

To show the performance improvement with the conditional servocompensator, we consider a second-order system modelled by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\theta_1(x_1 - x_1^3/3!) + \theta_2 u, \quad y = x_1 \tag{23}$$

with the reference signal $r(t) = r_0 \sin(\omega t)$, which is generated by the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} w, \quad w^{\mathrm{T}}(0) = [0, r_0], \quad r(t) = w_1$$

It can easily be verified that

$$\chi = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1(w_1 - w_1^3/3!)]$$

and that χ satisfies the identity $L_s^q \chi = c_0\chi + c_1 L_s\chi + \cdots + c_{q-1}L_s^{q-1}\chi$ with $q = 4, c_0 = -9\omega^4, c_1 = 0, c_2 = -10\omega^2$ and $c_3 = 0$. We show the performance of four designs: the first is an ideal SMC, the second is a continuous approximation that does not use a servocompensator, the third uses the fourth-order conventional servocompensator $\dot{\sigma} = S\sigma + Je_1$, and the last design uses the conditional servocompensator (13). In the first two designs, $\hat{s} = k_1 e_1 + \hat{e}_2$, while

$s = K_1\sigma + k_1 e_1 + \hat{e}_2$ in the last two designs. For all designs except the conventional servocompensator, the scalar k_1 is chosen as any positive constant. In the conditional servocompensator design, K_1 is chosen to make $(S - JK_1)$ Hurwitz. For the conventional servocompensator, k_1 and K_1 are chosen to make the matrix

$$\mathscr{A}_h = \begin{bmatrix} S & J \\ -K_1 & -k_1 \end{bmatrix}$$

Hurwitz (see Reference [17]). The estimate $\hat{e}_2$ is provided by the high-gain observer

$$\dot{\hat{e}}_1 = \hat{e}_2 + g_1(e_1 - \hat{e}_1)/\varepsilon, \quad \dot{\hat{e}}_2 = g_2(e_1 - \hat{e}_1)/\varepsilon^2$$

with g_1 and g_2 chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. The control is taken as $u = -k\,\text{sat}(\hat{s}/\mu)$.

We use the following numerical values in the simulation: $\theta_1 = 1, \theta_2 = 3, \omega = 0.5\,\text{rad/s}, r_0 = 1$, $k = 10$, $\mu = 0.1$, $k_1 = 5$ in the first, second and last designs, with K_1 chosen to assign the eigenvalues of $S - JK_1$ at $-0.5, -1, -1.5$ and -2. For the third design, we choose k_1 and K_1 to

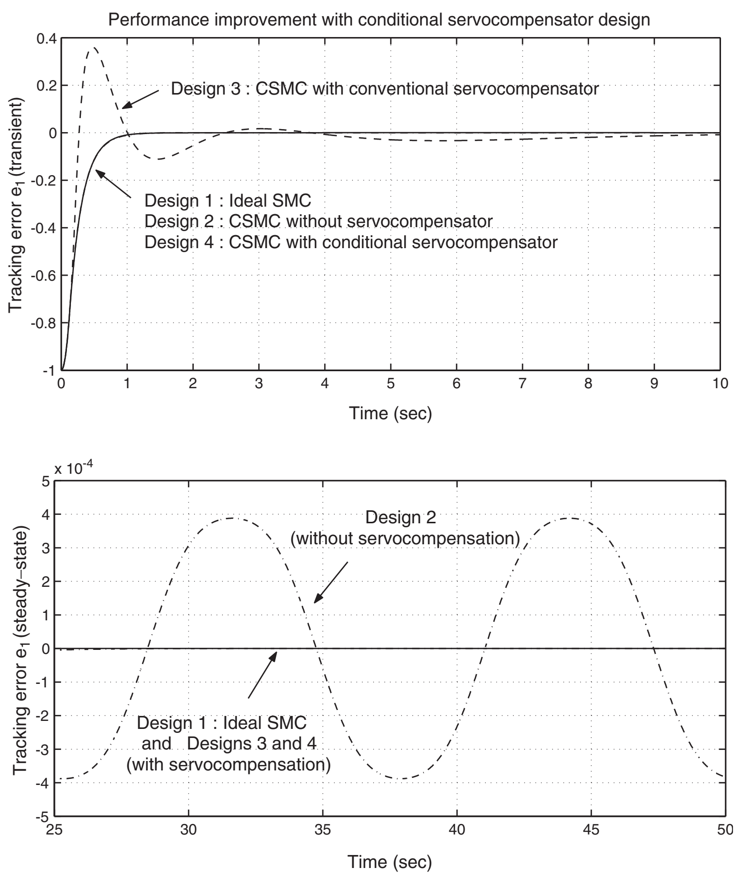

Figure 1. Performance improvement over the conventional servocompensator design using a conditional servocompensator.

assign the eigenvalues of $\mathscr{A}_h$ at $-0.5, -1, -1.5, -2$ and -3. The observer parameters are chosen as $g_1 = 6$, $g_2 = 5$ and $\varepsilon = 0.01$. The results of the simulation are shown in Figure 1, and the improvement in the transient performance with the conditional servocompensator is clear. In particular, the transient response of this design is close (indistinguishable in the figure) to that of the ideal SMC design. As expected, the transient response of the CSMC design without a servocompensator is also close to that of the ideal SMC, but does not result in asymptotic error convergence, while the conditional servocompensator design does. Zero steady-state error is also achieved with the third design, which employs a conventional servocompensator, but at the expense of degraded transient performance. Equation (23) represents an approximation of the pendulum equation. To show the effect of internal model perturbations, suppose that the system in question is the simple pendulum, described by the equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\theta_1 \sin(x_1) + \theta_2 u, \quad y = x_1 \tag{24}$$

With the control objective the same as that in the previous simulation, it can be verified that, in the current case

$$\chi = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1 \sin(w_1)]$$

so that Assumption 5 does not hold. Suppose that $\sin(w_1)$ is approximated by the successively higher order polynomials $p_1(w_1) = w_1$, $p_2(w_1) = w_1 - w_1^3/3!$, and $p_3(w_1) = w_1 - w_1^3/3! + w_1^5/5!$, respectively, to which correspond the perturbed nominal values of the steady-state control

$$\bar{\chi}_i = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1 p_i(w_1)], \quad i = 1, 2, 3$$

It can be verified that $\bar{\chi}_1$ satisfies (6) with $q = 2$, $c_0 = -\omega^2$, and $c_1 = 0$, while $\bar{\chi}_3$ does so with $q = 6$, $c_0 = -225\omega^6$, $c_1 = 0$, $c_2 = -259\omega^4$, $c_3 = 0$, $c_4 = -35\omega^2$, and $c_5 = 0$. The constants for $\bar{\chi}_2$ are as specified in the previous simulation. We compare the performance of three conditional servocompensator designs, of orders 2, 4, and 6, corresponding to the polynomial approximations $p_1(\cdot)$, $P_2(\cdot)$, and $p_3(\cdot)$, respectively. For the servocompensator of order 2, K_1 is chosen to assign the eigenvalues of $S - JK_1$ at -0.5 and -1, for that of order 4, at $-0.5, -1, -1.5$ and -2, and for that of order 6, at $-0.5, -1, -1.5, -2, -2.5$ and -5. All other values are retained from the previous simulation, except k, which is chosen as 20. The results are shown in Figure 2(a). For comparison, we also show the performance of the conventional servocompensator design of Khalil [17], with the eigenvalues of $\mathscr{A}_h$ placed as in Reference [17]. As expected from the results of Theorem 3, for both designs, there is a reduction in the steady-state tracking error going from the lowest order approximation to the highest. Figure 2(b) shows the transient response of the controllers. We see that while the transient responses are almost identical for the three designs in the case of the conditional servocompensator (indistinguishable in the figure), they become progressively degraded as the order of the approximation increases in the case of the conventional servocompensator design.

7. CONCLUSIONS

We have presented a new approach to the regulation of minimum-phase nonlinear systems. In the new approach, servocompensation is provided only 'conditionally', i.e. inside the boundary layer of a sliding mode control, thus effectively eliminating the transient performance

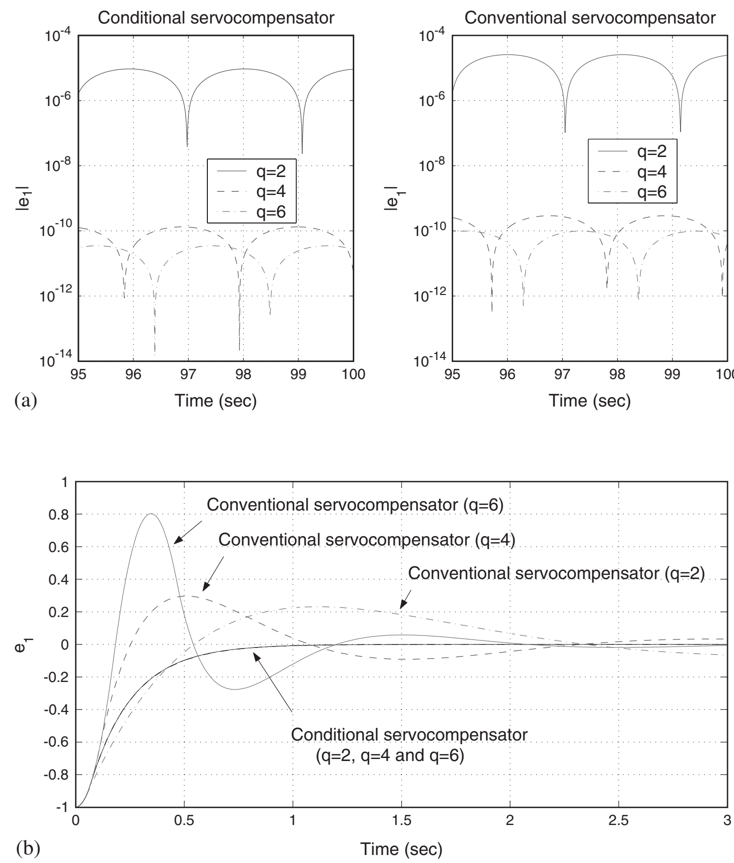

Figure 2. Effect of internal model perturbation on the tracking error: (a) steady-state tracking error $|e_1|$ (absolute value); and (b) tracking error e_1 during the transient period.

degradation brought about by the conventional servocompensator design. Analytical results are provided for regional and semi-global asymptotic tracking, and the improvement in performance is shown analytically by proving that the performance of the output feedback continuous sliding mode controller, with a conditional servocompensator, can be tuned to recover the performance of an ideal state feedback sliding mode controller, without a servocompensator.

We also studied the effect of internal model perturbations on the tracking error, and showed that in the presence of perturbation, the tracking error is ultimately bounded, with a bound that depends on the magnitude of the perturbation. In the case of such perturbations resulting from the approximation of a continuous function by polynomials, the magnitude of the perturbation can be made arbitrarily small, by increasing the order of the approximating polynomial. However, doing so increases the order of the internal model, and hence the system order. While in general, the transient response of a system becomes worse as its order increases, such is not the case with the conditional servocompensator. In particular, our result shows that the performance with the conditional servocompensator is always ‘close’ to that with an ideal sliding mode controller of fixed order, regardless of the order of the conditional servocompensator. This shows as an advantage of the conditional servocompensator design over the conventional one, in which the transient response becomes progressively degraded as the order of the approximating internal model increases.

Extensions to relax the design to include an equivalent control component and allow the coefficient of the switching component to be error and/or time dependent should be straightforward, as should be extensions to the multi-input, multi-output case. Such extensions are carried out in the special case of integral control in Reference [22].

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